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LETTER TO THE EDITOR

Cantori for symplectic maps

Qi Chen†¶, R S MacKay‡ and J D Meiss§||

† Institute for Fusion Studies, University of Texas, Austin, TX 78712, USA
‡ Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK
§ Lawrence Berkeley Laboratory, University of California, Berkeley, CA 94720, USA

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Abstract. We construct invariant sets for the $2d$-dimensional generalization of the sawtooth map which are semi-conjugate to any incommensurate rotation vector. When $d \leqslant 2$ we show that these are Cantor sets. These invariant sets are hyperbolic, and we give a structural stability argument to show the existence of cantori for a non-trivial class of smooth symplectic maps.

Aubry and Mather [1, 2] show that orbits with any rotation number exist for area-preserving twist maps. Extension of this theory to symplectic maps with more than one degree of freedom appears to be a difficult task. Bangert [3] and Mather [4] have results about the set of rotation vectors for orbits of minimal action; however, an example of Hedlund [5] shows that in general one cannot hope to obtain minimizing orbits with all rotation vectors. Bernstein and Katok [6] show that for maps close enough to integrable, the minimizing periodic orbits satisfy some regularity properties which are sufficient to allow the existence of a limiting orbit as the rotation vector approaches any limit. However, they cannot prove anything about the rotation vectors of the limiting orbits.

Here we consider the opposite situation, when the 'potential' dominates the 'kinetic' energy. In this limit, the minimizing orbits of an area-preserving map approach the minimizing orbits of the 'sawtooth mapping' which is a piecewise linear, discontinuous mapping. As was shown independently by Aubry and Percival [7–10], an explicit formula can be obtained for orbits of the sawtooth mapping with irrational rotation number; these are dense on Cantor sets, and are called cantori. We consider the generalization of the sawtooth mapping to d degrees of freedom, and obtain a formula for a set of orbits with incommensurate rotation vectors. When $d \leqslant 2$, these are shown to cover Cantor sets.

We use the structural stability of hyperbolic sets to show that the cantori persist under perturbations of the mapping which can be arbitrary in a region around the discontinuity, but which are C^1 small on the cantorus.

Let $x = (x_1, x_2, \ldots, x_d)$ represent a configuration point in $\mathbb{R}^d$. Consider a mapping with the generating function

$$h(x, x') = \tfrac{1}{2}|x - x'|^2 + \frac{\lambda}{2}\{x\}^{\mathrm{t}} Q\{x\} \tag{1}$$

|| Permanent address: Department of Applied Mathematics, University of Colorado, Boulder, CO 80309, USA.

¶ Present address: Physics Department, University of Maryland, College Park, MD, USA.

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where $x, x' \in \mathbb{R}^d$, Q is a positive definite matrix, $\lambda > 0$, superscript t represents transpose, and $\{x\}$ represents a fractional part of x with respect to the group of translations by integer vectors, $\mathbb{Z}^d$. For definiteness, choose some $\sigma = (\sigma_1, \ldots, \sigma_d)$ with $\sigma_j \in \{+, -\}$ and define the fractional part

$$\{x\}^{\sigma_i} \equiv x_i - [x_i + \tfrac{1}{2}]^{\sigma_i} \tag{2}$$

where $[x]^+$ ($[x]^-$) is the right (left) continuous integer part. A function $f(x)$ for which $f(x + \sigma\varepsilon) \to f(x)$ as $\varepsilon \to 0$ for all x and $\varepsilon > 0$, will be called *σ-continuous.* The fractional part $\{x\}^\sigma$ is σ-continuous.

A trajectory generated by (1) consists of a configuration $(x^s, x^{s+1}, \ldots, x^t)$, $s < t$ which is a stationary point of the action

$$W[x^s, x^{s+1}, \ldots, x^t] = \sum_{j=s}^{t-1} h(x^j, x^{j+1}).$$

Differentiation yields the second difference equation

$$x^{t+1} - 2x^t + x^{t-1} = \lambda Q\{x^t\} \tag{3}$$

provided none of the configuration points falls on the discontinuities. This defines a map $S : (x^{t-1}, x^t) \to (x^t, x^{t+1})$ which is the $2d$-dimensional generalization of the sawtooth mapping [10]. If Q is diagonal the components of (3) decouple and it can be treated as a set of d area-preserving sawtooth mappings; however, in general, (3) cannot be decoupled.

Consider an incommensurate rotation vector $\omega \in \mathbb{R}^d$ (i.e. $k \cdot \omega \notin \mathbb{Z}\ \forall k \in \mathbb{Z}^d \backslash 0$). We look for invariant sets with this rotation vector given by a function $x : \mathbb{R}^d \to \mathbb{R}^d$ where $x(\theta + k) = x(\theta) + k$, $\forall k \in \mathbb{Z}^d$, and such that the orbit of $x(\theta_0)$ is $x(\theta_0 + \omega t)$, $\forall \theta_0 \in \mathbb{R}^d$ and $t \in \mathbb{Z}$. By (3), $x(\theta)$ must satisfy

$$x(\theta + \omega) - 2x(\theta) + x(\theta - \omega) = \lambda Q\{x(\theta)\}. \tag{4}$$

Such sets exist if the parameter λ is large enough as is shown by the following.

Proposition. There exists an $L > 0$ such that for $\lambda > L$ and any incommensurate $\omega \in \mathbb{R}^d$, there is an invariant set $\tilde{M}_\omega$ of the sawtooth map of the form

$$\tilde{M}_\omega = \{(x^\sigma(\theta), x^\sigma(\theta + \omega)) \mid \theta \in \mathbb{R}^d, \sigma \in \{+, -\}^d\} \tag{5}$$

such that $x^\sigma(\theta + k) = x^\sigma(\theta) + k\ \forall k \in \mathbb{Z}^d$ and that $x^\sigma(\theta_0 + \omega t)$ is an orbit for all $\theta_0 \in \mathbb{R}^d$ and $\sigma \in \{+, -\}^d$. The solution is given by

$$x^\sigma(\theta) = \theta - \sum_n B^{-1} b(n) B\{\theta + n\omega\}^\sigma \tag{6}$$

where B is an orthogonal matrix which diagonalizes Q, and $b(n)$, $n \in \mathbb{Z}$, are diagonal matrices of the form

$$b(n) = \mathrm{diag}(\alpha_i \rho_i^{-|n|}). \tag{7}$$

with α_i and ρ_i positive (see (8)).

Proof. We first show that a solution to (4) can be found of the form (6) under the assumptions that the solution satisfies $[x^\sigma(\theta)] = [\theta]$ and that $x(\theta)$ has no points on the discontinuity set of $\{\ \}$. Let $x^\sigma(\theta)=\theta+\psi(\theta)$; by assumption $\{x^\sigma(\theta)\}$ is independent of the choice of fractional part, so choose the σ-continuous one:

$$\begin{aligned}\{x^\sigma(\theta)\}&=x^\sigma(\theta)-[x^\sigma(\theta)]^\sigma\\&=\theta+\psi(\theta)-[\theta]^\sigma\\&=\{\theta\}^\sigma+\psi(\theta).\end{aligned}$$

Therefore by (4) ψ must satisfy

$$\psi(\theta+\omega)-2\psi(\theta)+\psi(\theta-\omega)=\lambda Q[\{\theta\}^\sigma+\psi(\theta)].$$

A solution to this can be obtained from the ansatz

$$\psi(\theta)=\sum_{n=-\infty}^{\infty}a(n)\{\theta+n\omega\}^\sigma$$

providing

$$\begin{aligned}&a(n-1)-2a(n)+a(n+1)=\lambda Qa(n)\qquad n\neq 0\\&a(-1)-2a(0)+a(1)=\lambda Q(I+a(0)).\end{aligned}$$

Since Q is symmetric, it can be diagonalized by an orthogonal coordinate change. Let $Q=B^{-1}DB$, where $B^t=B^{-1}$ and $D=\mathrm{diag}(q_i)$ is a diagonal matrix; since Q is positive definite, the entries q_i are positive reals. Defining $b(n)=-Ba(n)B^{-1}$, then the solution for $b(n)$ which decays as $n\to\pm\infty$ is given by (7) providing

$$\begin{aligned}\rho_i&=1+\frac{\lambda q_i}{2}+\left(\lambda q_i+\frac{(\lambda q_i)^2}{4}\right)^{1/2}\\\alpha_i&=\left(1+\frac{4}{\lambda q_i}\right)^{-1/2}.\end{aligned}\tag{8}$$

So we obtain $x(\theta)$ of the form (6) providing the assumptions $[x^\sigma(\theta)]=[\theta]$, and $x^\sigma(\theta)$ has no points on the discontinuity set are satisfied. To verify this we note the following properties of (6).

(*a*) $x^\sigma(\theta)$ is σ-continuous. This follows because for $\varepsilon>0$, $x^\sigma(\theta+\sigma\varepsilon)\to x^\sigma(\theta)$ as $\varepsilon\to 0$, since $\{\theta\}^\sigma$ is σ-continuous.

(*b*) The derivative of $x^\sigma(\theta)$ vanishes at all points of continuity. Since the series for $x(\theta)$ is uniformly convergent, its derivative can be computed term by term. Evaluating the derivative of (6) at a point where $\{\theta+n\omega\}^\sigma$ is continuous $\forall n$ gives

$$\frac{\partial x}{\partial\theta}=I-B^{-1}\sum_n b(n)B=0$$

which follows from the sum

$$\sum_n\rho_i^{-|n|}=\frac{\rho_i+1}{\rho_i-1}=\frac{1}{\lambda q_i}\left(\rho_i-\frac{1}{\rho_i}\right)=\frac{1}{\alpha_i}.\tag{9}$$

Thus all changes in $x^\sigma(\theta)$ occur in the discontinuities of $\{\theta+n\omega\}^\sigma$, that is the points

$$\theta_j+n\omega_j=m-\tfrac{1}{2}$$

for any integer m. Such points are dense.

(*c*) $x_j^\sigma(\theta)$ is a monotonic function of θ_j. The jump in x^σ across a point $\theta_j=m-n\omega_j-1/2$, holding all the other θ_i, $i\neq j$, fixed, is

$$\Delta x(n;j)=B^{-1}b(n)B\,e^j\qquad\forall\theta_i,\ i\neq j\text{ and }\sigma\tag{10}$$

where e^j is the unit vector in the j direction. Thus $\langle e^j, \Delta x(n; j)\rangle > 0$, so x_j^σ strictly increases across such a discontinuity. Since the discontinuities are dense, x_j^σ increases between almost every pair of values of θ_j.

(*d*) $x(\theta)$ can be written as a sum of monotone vectors, each depending on a single θ_j:

$$\begin{aligned} x^\sigma(\theta) &= \sum_{j=1}^{d} x^j(\theta_j) \\ x^j(\theta_j) &= \theta_j - B^{-1} \sum_n b(n)\{\theta_j + n\omega_j\}^{\sigma_i} B\, e^j. \end{aligned} \tag{11}$$

We call a vector function $x^j(\theta_j)$ monotone if $\langle e^j, x^j(\theta_j+\delta) - x^j(\theta_j)\rangle \geqslant 0$, $\forall \delta \geqslant 0$. This is true for (11) as follows from (10).

A bound on the norm of $x^j(\theta_j)$ can be obtained by noting that $x^j(0) = 0$ and at any other point its value is the sum of the jumps $\Delta x(n; j)$ for those n corresponding to discontinuities in $\{\varphi + n\omega_j\}$ for $\varphi \in [0, \theta_j]$. Providing $|\theta_j| < 1/2$, the discontinuity corresponding to $n = 0$ does not contribute. Furthermore $x^j(\theta_j)$ is odd almost everywhere, indeed $x^j(\theta_j + \varepsilon) \to -x^j(-\theta_j - \varepsilon)$ as $\varepsilon \to 0^+$, so only half of the remaining discontinuities contribute, thus

$$|x^j(\theta_j)| \leqslant \frac{1}{2} \sum_{n\neq 0} |\Delta x(n; j)| \leqslant \frac{1}{2} \sum_{n\neq 0} \sum_{i=1}^{d} \alpha_i \rho_i^{-|n|} = \frac{1}{2} \sum_{i=1}^{d} (1 - \alpha_i).$$

Thus from (11) the norm of $x(\theta)$ itself is bounded by

$$|x(\theta)| \leqslant \frac{d}{2} \sum_{i=1}^{d} (1 - \alpha_i) \qquad \text{for } |\theta_j| < \tfrac{1}{2}. \tag{12}$$

This implies $x(\theta)$ is certainly in the fundamental domain $(-\frac{1}{2}, \frac{1}{2})^d$ when θ is, providing λ satisfies

$$\lambda > L = \frac{4}{\min(q_i)} \left(\frac{d^4}{2d^2 - 1} - 1 \right) \tag{13}$$

which completes the proof. □

An example for $d = 2$ is shown in figure 1. Here we chose $\lambda q = (0.02, 0.03)$, and let B be a rotation by an angle 0.5. The rotation vector is $\omega = ((3 - \sqrt{5})/2, \sqrt{2} - 1)$. Values for θ were taken on a rectangular grid.

For $d = 1$, (13) implies that the solution (6) is valid for all $\lambda > 0$. For $d > 1$, this is not necessarily the case, and numerical calculation for $d = 2$ shows that if the q_i are too small then $x^\sigma(\theta)$ can cross the discontinuity and therefore will not represent a valid solution. In practice the bound (13) is extremely weak; for $d = 2$ the set appears to remain in the fundamental domain until $\lambda \min(q_i) \sim O(10^{-3})$. Even in this case, however, it is possible to modify the fundamental domain $(-\frac{1}{2}, \frac{1}{2})^d$ used in the definition of $\{x\}$ to obtain a mapping for which (6) is valid for any λ †.

Note that (6) also gives invariant sets with commensurate rotation vector, whose topological form is different; in this letter we concentrate on the incommensurate case.

We now consider the topological form of the set $\tilde{M}_\omega$. The sawtooth map commutes with the translations $(x, x') \to (x + m, x' + m)$, for $m \in \mathbb{Z}^d$, so it can be considered as a map of $\mathbb{R}^d \times \mathbb{R}^d / \mathbb{Z}^d$ to itself.

† This follows from the proposition below: choose the boundaries of the fundamental domain to run in the gaps between the $\gamma^1(0, \pm)$ curves and the $\gamma^2(0, \pm)$ curves.

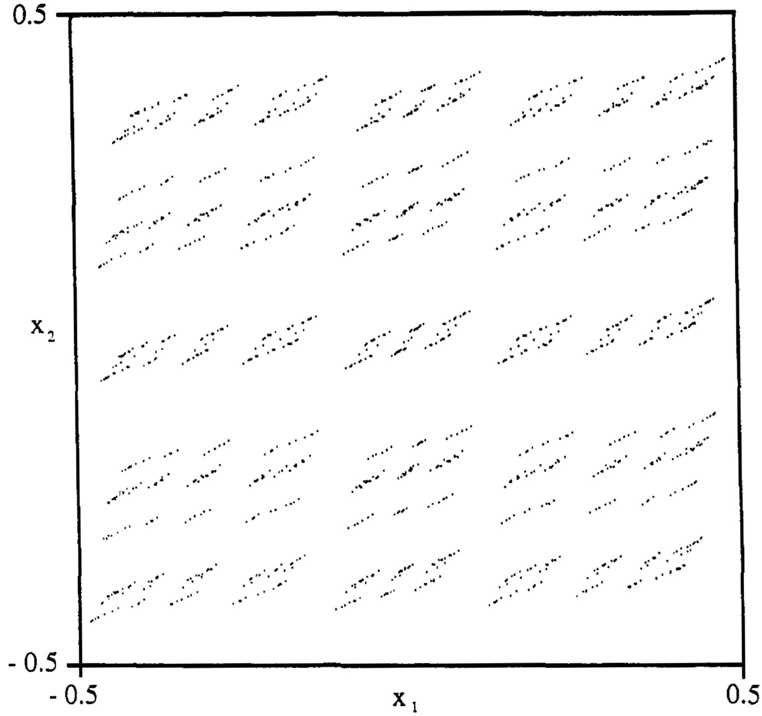

Figure 1. Configuration space projection of a cantorus of the four-dimensional sawtooth map.

Proposition. Let ω be incommensurate and $M_\omega = \tilde{M}_\omega/\mathbb{Z}^d$, with $\tilde{M}_\omega$ given by (5). Then for $d \leqslant 2$, M_ω is a Cantor set.

Proof. A Cantor set is a topological space which is non-empty, compact, totally disconnected and has no isolated points. Since ω is incommensurate, it follows from property (a) above that for all d, M_ω is non-empty, closed and has no isolated points. Furthermore, since the $b(n)$ are summable, M_ω is bounded. So it remains to prove that M is totally disconnected.

For $d=1$, this follows from the fact that the jumps (10) are positive and that ω is incommensurate.

For $d=2$, consider a point $\theta_j = n_j\omega_j + 1/2$, $j=1,2$. The vectors $\Delta x(n_j; j)$ given by (10) span an area of positive orientation. This area is the same as that spanned by $B\Delta x(n_j; j)$ since $B \in \mathrm{SO}(2)$. Letting φ be the rotation angle, this area is given by the cross product

$$B\Delta x(n_1; 1) \times B\Delta x(n_2; 2) = \alpha_1\alpha_2[\rho_1^{-|n_1|}\rho_2^{-|n_2|}\cos^2\varphi + \rho_2^{-|n_1|}\rho_1^{-|n_2|}\sin^2\varphi] > 0. \qquad (14)$$

Define two curves $\gamma^1(j, \sigma_1)$ by joining the gaps in the points of $x^\sigma(j\omega_1 + 1/2, \theta_2)$, for $\theta_2 \in \mathbb{R}$ and $\sigma_2 \in \{+,-\}$ with straight lines. By (c) these curves are graphs over the x_2 axis. Similarly define the graphs $\gamma^2(j, \sigma_2)$ over x_1. The positivity of (14) for all n_2 implies that $\gamma^1(n_1, +)$ and $\gamma^1(n_1, -)$ do not intersect, and similarly for γ^2. Then (c) implies that the four curves $\gamma^i(n_i, \pm)$, as shown in figure 2, separate the points of M_ω with $\theta_i > n_i\omega_i$ and $\theta_i < n_i\omega_i$ where $i = 1,2$.

Since ω_1 and ω_2 are irrational, this implies that M_ω is totally disconnected, thus it is a Cantor set. $\square$

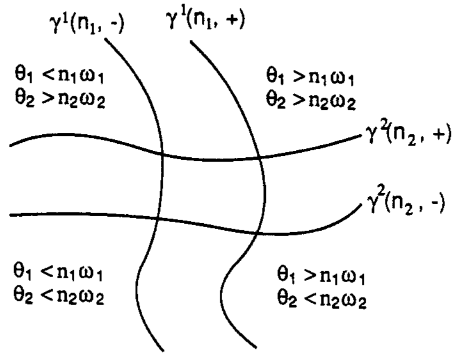

Figure 2. Disconnectedness of the set M_ω.

We believe that this result is also true for $d>2$, but did not succeed in proving it.

We now show that many smooth symplectic maps have invariant sets topologically equivalent to those found for the sawtooth map. Although the sawtooth map is discontinuous, the invariant sets M_ω avoid a neighbourhood of the discontinuity set so the map can be modified there in any way leaving M_ω unchanged. In particular, there are C^∞ symplectic maps with the same M_ω. Since the invariant sets of the sawtooth map are hyperbolic, with hyperbolicity constant $\min(\rho_i)$, they are structurally stable, that is M_ω persists for all perturbations of the map which are C^1 small enough in a neighborhood of M_ω; see, e.g., [11]. This allows us to deduce existence of topologically equivalent invariant sets for an open set of smooth symplectic maps of $\mathbb{R}^d \times \mathbb{R}^d/\mathbb{Z}^d$. This set includes non-trivial examples, as in the following.

Theorem. Suppose f is a symplectic map with generating function $h(x,x')=\frac{1}{2}|x-x'|^2+\lambda V(x)$, $x\in\mathbb{R}^d$, such that $V(x+k)=V(x)\;\forall k\in\mathbb{Z}^d$, and V has a non-degenerate local minimum with $D^3V(x)=0$ at this minimum (e.g. because of symmetry). Then there exist invariant sets topologically equivalent to M_ω for all ω, provided λ is large enough.

Proof. Without loss of generality let $x=0$ be the minimum of V; and write the quadratic part of V there as $\frac{1}{2}x^tQx$. For θ in the fundamental domain, (12) implies that the invariant set M_ω of the sawtooth map is contained within the region R: $\{|x|<d^2/\lambda\min(q_i)\,|\,x\in\mathbb{R}^d\}$. In this region $\lambda V(x)$ is C^2 close to its quadratic part, and so f is C^1 close to the sawtooth mapping. Hence M_ω persists for large enough λ. □

The above example can be modified to $h(x,x')=T(x-x')+\lambda[V(x)+W(x')]$, where T represents any positive definite quadratic form (paper in preparation). We believe that the restriction $D^3V=0$ is not necessary, since f would still be C^0 close to the sawtooth map, S, on R and the relative change in the derivative $(\mathrm{D}f\,DS^{-1}-I)$ would be small.

There are several remaining questions.

(1) Is M_ω a Cantor set when $d>2$? We suspect that this is the case. In particular, since M_ω is hyperbolic and its orbits are semi-conjugate to a rotation, it has zero Hausdorff dimension [12].

(2) Can the theorem be generalized to any $V(x)$ with a non-degenerate minimum for which $D^3V \neq 0$? Can one generalize to examples with a more general dependence of h on (x, x')?

(3) Do these orbits globally minimize the action? It is clear that they are local minima; however, we have not shown that variations which move a configuration point across the discontinuity necessarily do not decrease the action. Even if M_ω consists of orbits of minimum action for the sawtooth do the perturbed sets for nearby maps consists of minimizing orbits? For $d = 1$, Mather has given examples where the continuation of a hyperbolic Aubry–Mather set by structural stability no longer consists of minimizing orbits [13].

(4) Can these Cantor sets be regarded as the ghosts which remain when invariant tori break up? Does every invariant torus have such a ghost, or something similar? In figure 3 we display a periodic orbit of the four-dimensional Froeschlé mapping [14] with primitive period 78635. When the parameter λ is small this orbit appear to nearly uniformly cover the fundamental domain; it is a close approximation to an invariant torus with incommensurate frequency. As λ increases the density of points becomes non-uniform, and low density regions begin to form. It appears that when λ is larger these low density regions will be empty, and the orbit will cover a Cantor set. Unfortunately in this situation the orbit becomes highly unstable, and impossible to follow numerically.

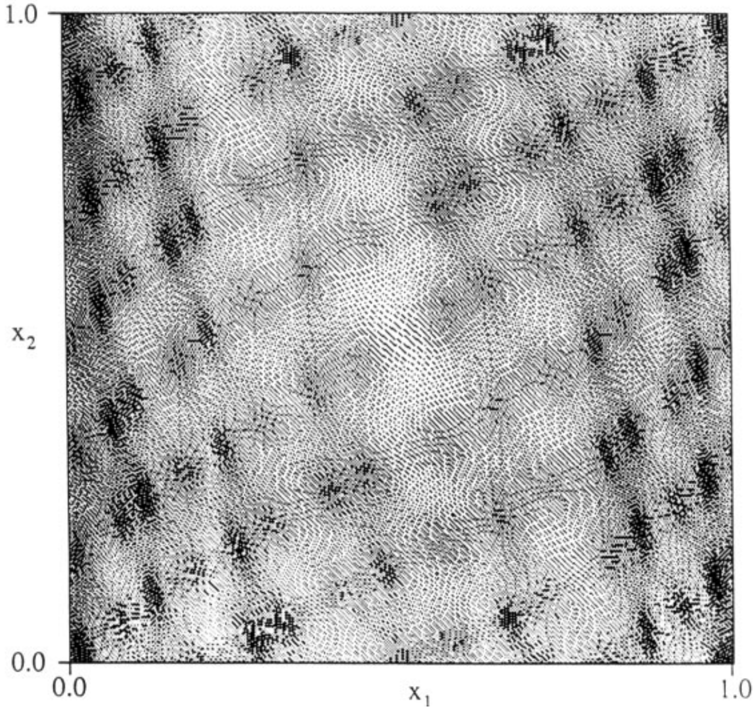

Figure 3. Configuration space projection of a periodic orbit with rotation vector (17556/78635, 51016/78635) for the four-dimensional Froeschlé map with parameters $(a, b, c) = (0.5, 0.35, 0.02)$, corresponding to $\lambda q \approx (0.483, 0.327)$.

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